Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 29 (2008) 1298–1308
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Extended set partitions with successions

Augustine O. Munagi

The John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand, Johannesburg 2050, South Africa

Received 2 October 2006; accepted 4 June 2007
Available online 30 July 2007

Abstract

We enumerate set partitions by strings of consecutive elements, or successions, and obtain a formula for the number of partitions with successions of arbitrary length. Our approach involves direct operations on the objects within the blocks of partitions. The succession concept is extended to m-regular partitions by means of two algorithms for transforming partitions. We also present a succession-based connection between integer partitions and set partitions, and obtain an application to the enumeration of partitions of arbitrary subsets of $\{1, 2, \ldots, n\}$ by successions.

1. Introduction

A partition of $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty disjoint subsets of $[n]$, called blocks, whose union is $[n]$. The number of all k-partitions of $[n]$ is the Stirling Number of the second kind, to be denoted by $S(n, k)$.

Given integers $x > 0$, $t > 1$, a t-succession is defined as the t numbers $x, x+1, \ldots, x+t-1$. A partition p of $[n]$ is said to contain a t-succession if a block of p does. Denote the set of k-partitions of $[n]$ with r occurrences of t-successions by $C_t(n, k, r)$, $r \geq 0$, and let the cardinality $|C_t(n, k, r)|$ be represented by $c_t(n, k, r)$.

Our first result is an explicit general formula for $c_t(n, k, r)$, $t \geq 2$ (Theorem 3). The derivation will rely on two special subsets $D_t(n, k, r)$ and $W_t(n, k, r)$ of $C_t(n, k, r)$ defined as follows.
$p \in D_t(n, k, r)$ if $p \in C_t(n, k, r), r > 1$, and p contains only u-successions, where $u = t$.
$p \in W_t(n, k, r)$ if $p \in C_t(n, k, r), r > 1$, and p contains only u-successions, where $u \in [t]$. Thus $D_t(n, k, r) \subseteq W_t(n, k, r) \subseteq C_t(n, k, r)$, $D_2(n, k, r) = W_2(n, k, r)$, and $W_t(n, k, r) = C_t(n, k, r)$, for $(t, r) = (1, r), (t, 1), (t, 0)$.

E-mail address: munagi@maths.wits.ac.za.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2007.06.003

For example, two elements of $C_3(n,k,r)$ are shown after each containing subset below:
$D_3(10,3,2)$: (1,2,3,5,7)(4,8,9,10)(6), (1,10)(2,6)(3,4,5,7,8,9);
$W_3(10,3,2)\backslash D_3(10,3,2)$: (1,2,3)(4,5,7,8,9)(6,10), (1,2,6)(3,4,5,7,8,9)(10);
$C_3(10,3,2)\backslash W_3(10,3,2)$: (1,2,3,4,10)(3,7,9)(6,8), (1,2,4,5)(3,10)(6,7,8,9).
We will use the notations $|D_t(n,k,r)| = d_t(n,k,r)$, $|W_t(n,k,r)| = w_t(n,k,r)$.
The results in the following theorem are established in [4].

Theorem 1.

(i) $c_2(n,k,r) = \binom{n-1}{r} S(n-r-1,k-1)$.

(ii) $d_t(n,k,r) = \binom{n-(t-1)r}{r} S(n-(t-1)r-1,k-1)$.

Note that Theorem 1(ii) implies the identity

$$d_t(n,k,r) = d_{t+j}(n+jr,k,r), \quad j = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

In Section 3 we extend the succession concept to the set $H^m(n,k)$ of m-regular k-partitions of $[n]$, $m \geq 1$. A partition p of $[n]$ is m-regular if any pair of distinct elements x, y in a block satisfies $|x-y| \geq m$. If all the blocks are singletons then p is N-regular, $N \geq n$. The corresponding extension of Theorem 1(i) is given in Theorem 4. We also examine a bijection between $H^{m+1}(n+1,k+1)$ and $H^m(n,k)$, which is due to Chen et al. [2], and a special bijection between $H^1(n,k)$ and $H^2(n+1,k+1)$ the generalization of which is given in Theorem 5.

Section 4 gives a succession-based relation between set partitions and integer partitions which seems to be new (Theorem 6). Lastly, Section 5 broaches the general problem of enumerating the partitions of any subset of $[n]$ containing a given number of successions.

2. Partitions with t-successions

In this section we derive the formula for $c_t(n,k,r)$, when $t > 2$. Let $w_t(n,k,r_t,r_{t-1},\ldots,r_x)$ denote the number of partitions enumerated by $w_t(n,k,r_t)$ such that each partition contains r_j j-successions, $r_j > 0$, $2 \leq x \leq t-1$. It follows that

$$w_t(n,k,r_t,r_{t-1},\ldots,r_{x+1}) = \sum_{0 \leq r_x \leq \triangle_x} w_t(n,k,r_t,r_{t-1},\ldots,r_x), \tag{2}$$

where $\triangle_x = \lfloor \frac{n-tr_t-(t-1)r_{t-1}\cdots-(x+1)r_{x+1}}{x} \rfloor$, $2 \leq x \leq t-1$, and $\lfloor N \rfloor$ is the floor function.

Lemma 1. *Let $r_t, r_{t-1}, \ldots, r_x$ be fixed nonnegative integers, $2 \leq x \leq t$. Then*

$$w_t(n,k,r_t,r_{t-1},\ldots,r_x) = \frac{(r_t+r_{t-1}+\cdots+r_x)!}{r_t!r_{t-1}!\ldots r_x!} \times w_x(n-(t-x)r_t-(t-1-x)r_{t-1}-\cdots-r_{x+1},k,r_t+r_{t-1}+\cdots+r_x).$$

Proof. Starting with a partition enumerated by $w_t(n,k,r_t,r_{t-1},\ldots,r_x,0,\ldots,0)$ each j-succession ($j \leq t-1$) can be transformed into a t-succession by appending the $t-j$ integers $v+1, v+2, \ldots, v+t-j$, where v is the greatest member of the succession, and then adding $t-j$ to every integer elsewhere which is greater than v.

Consequently each group of the r_j j-successions, following extensions to t-successions, increases n by $(t-j)r_j$. Thus after transforming all the successions to t-successions we obtain the term

$$d_t(n + r_{t-1} + 2r_{t-2} + \cdots + (t-x)r_x, k, r_t + r_{t-1} + \cdots + r_x). \tag{3}$$

Conversely, to obtain a partition enumerated by $w_t(n, k, r_t, r_{t-1}, \ldots, r_x, 0, \ldots, 0)$ from a partition q enumerated by (3) we must successively select r_j t-successions and reduce them to j-successions, by deleting $t-j$ integers from each, and then subtracting $t-j$ from every integer greater than $v+t-j$. Thus the total number of ways of reconstructing a partition enumerated by $w_t(n, k, r_t, r_{t-1}, \ldots, r_x, 0, \ldots, 0)$ from q is, by the rule of product, equal to

$$\begin{aligned}&\binom{r_t + r_{t-1} + \cdots + r_2}{r_t}\binom{r_{t-1} + r_{t-2} + \cdots + r_2}{r_{t-1}}\cdots\binom{r_{x-1} + r_x}{r_{x-1}}\\ &\quad = \frac{(r_t + r_{t-1} + \cdots + r_x)!}{r_t! r_{t-1}! \ldots r_x!}.\end{aligned}$$

Hence, for all partitions enumerated by (3), we obtain the identity

$$\begin{aligned}&w_t(n, k, r_t, r_{t-1}, \ldots, r_x, 0, \ldots, 0)\\ &\quad = \frac{(r_t + r_{t-1} + \cdots + r_x)!}{r_t! r_{t-1}! \ldots r_x!} d_t(n + r_{t-1} + 2r_{t-2} + \cdots + (t-x)r_x,\\ &\qquad k, r_t + r_{t-1} + \cdots + r_x)\\ &\quad = \frac{(r_t + r_{t-1} + \cdots + r_x)!}{r_t! r_{t-1}! \ldots r_x!} d_x(n - (t-x)r_t - (t-1-x)r_{t-1} - \cdots - r_{x+1},\\ &\qquad k, r_t + r_{t-1} + \cdots + r_x),\end{aligned}$$

where the last equality follows from (1). The lemma then follows by extending both sides to include shorter-length successions. □

The first part of the next theorem is a consequence of (2) and Lemma 1, and the second part is obtained by iterating the first part (δ_{ij} is the Kronecker delta).

Theorem 2. (i) $w_1(n, k, r) = S(n-1, k-1)\delta_{nr}$, *and if* t *is an integer,* $t \geq 2$, *then*

$$w_t(n, k, r) = \sum_{j=0}^{\lfloor (n-tr)/(t-1) \rfloor} \binom{r+j}{r} w_{t-1}(n-r, k, r+j).$$

(ii) *For* $t > 2$, *we have*

$$\begin{aligned}w_t(n, k, r) = &\sum_{\substack{0 \leq r_2 \leq \triangle_2\\ \cdots\cdots\\ 0 \leq r_{t-1} \leq \triangle_{t-1}}} \frac{(r_t + r_{t-1} + \cdots + r_2)!}{r_t! r_{t-1}! \ldots r_2!}\\ &\times \binom{n - (t-1)r_t - (t-2)r_{t-1} - \cdots - r_2}{r_t + r_{t-1} + \cdots + r_2}\\ &\times S(n - (t-1)r_t - (t-2)r_{t-1} - \cdots - r_2 - 1, k-1),\end{aligned}$$

where $\triangle_j = \lfloor \frac{n - tr_t - (t-1)r_{t-1} \cdots - (j+1)r_{j+1}}{j} \rfloor$, $2 \leq j \leq t-1$.

Theorem 3.

$$c_t(n,k,r) = \sum_{v=1}^{r} \binom{r-1}{v-1} w_t(n-r+v,k,v), \quad r > 0. \tag{4}$$

$$c_t(n,k,0) = \sum_{j=0}^{\lfloor n/(t-1) \rfloor} w_{t-1}(n,k,j). \tag{5}$$

Proof. We employ the notation $(1^{r_1}, 2^{r_2}, \ldots)$ for an integer partition in which the part j appears r_j times, $r_j \geq 0$ [1, p. 122]. To prove (4) we consider the possible shapes of occurrences of the t-successions determined by the partitions of r such that a part in each partition represents the number of t-successions in a u-succession, $u \geq t$. Since a u-succession ($u \geq t$) contains exactly $u-t+1$ t-successions, the function $w_u(n,k,r_u,r_{u-1},\ldots,r_t)$ can be viewed as the enumerator of partitions in $C_t(n,k,(u-t+1)r_u + (u-t)r_{u-1} + \cdots + r_t)$ in which r_{u-i} $(u-i)$-successions contribute $(u-i-t+1)r_{u-i}$ t-successions for each i, $0 \leq i \leq u-t$. But the decomposition $(u-t+1)r_u + (u-t)r_{u-1} + \cdots + r_t$ represents the partition $(1^{r_t}, 2^{r_{t+1}}, \ldots)$ in which a part j appears r_{t+j-1} times, and the number of distinct parts is at most $u-t+1$. Thus the contribution of partitions of r into v parts to $c_t(n,k,r)$ is given by $A_v = \sum w_u(n,k,r_u,r_{u-1},\ldots,r_t)$, where the summation is over all the sequences $(r_t, r_{t+1}, \ldots, r_u, 0, \ldots)$ $(r_u > 0)$ of exponents of partitions of r into v parts, $1 \leq v \leq r$. So by Lemma 1 we have

$$\begin{aligned} A_v &= \sum \frac{(r_u + r_{u-1} + \cdots + r_t)!}{r_u! r_{u-1}! \ldots r_x!} w_t(n-(u-t)r_u - (u-1-t)r_{u-1} - \cdots - r_{t+1}, \\ &\quad k, r_u + r_{u-1} + \cdots + r_t) \\ &= \sum \frac{v!}{r_u! r_{u-1}! \ldots r_x!} w_t(n-r+v,k,v), \end{aligned}$$

since $(u-t)r_u + (u-1-t)r_{u-1} + \cdots + r_{t+1} = r - (r_u + r_{u-1} + \cdots + r_t) = r - v$. Hence $A_v = c(r,v)w_t(n-r+v,k,v)$, where $c(r,v) = \sum \frac{v!}{r_u! r_{u-1}! \ldots r_x!}$ is the number of compositions of the integer r into v parts. The result follows by summing over v, and applying the standard formula $c(n,k) = \binom{n-1}{k-1}$. Finally, (5) follows from the relation $c_t(n,k,0) = w_t(n,k,0)$ and Theorem 2(i). $\square$

Remark. As an illustration of the proof of Theorem 3 consider $r = 7$. The partitions of 7 into 4 parts namely $1+2+2+2 \equiv (1^1, 2^3, 3^0, \ldots)$, $1+1+2+3 \equiv (1^2, 2^1, 3^1, 4^0 \ldots)$, $1+1+1+4 \equiv (1^3, 2^0, 3^0, 4^1, 5^0 \ldots)$, give the following canonical sequences of exponents: $(1,3,0,\ldots)$, $(2,1,1,0,\ldots)$, $(3,0,0,1,0,\ldots)$. So the contribution A_4 of the partitions of 7 into 4 parts to $c_t(n,k,7)$ is given by

$$\begin{aligned} A_4 &= w_{t+1}(n,k,3,1) + w_{t+2}(n,k,1,1,2) + w_{t+3}(n,k,1,0,0,3) \\ &= 4w_t(n-3,k,4) + 12w_t(n-3,k,4) + 4w_t(n-3,k,4) \text{ (by Lemma 1)} \\ &= 20w_t(n-3,k,4) = \binom{6}{3} w_t(n-3,k,4), \text{ as desired.} \end{aligned}$$

Remarks. (i) It is easily verified that Theorem 3 gives the formula for $c_2(n,k,r)$ if $t = 2$.

(ii) If $t = 3$ in Theorem 2(ii), we obtain

$$w_3(n, k, r) = \sum_{j=r}^{\lfloor (n-r)/2 \rfloor} \binom{j}{r} \binom{n-r-j}{j} S(n-r-j-1, k-1).$$

Hence, Theorem 3 gives

$$c_3(n, k, r) = \sum_{v=1}^{r} \sum_{j=v}^{\lfloor (n-r)/2 \rfloor} \binom{r-1}{v-1} \binom{j}{r} \binom{n-r-j}{j} S(n-r-j-1, k-1).$$

Clearly the explicit formula for $c_t(n, k, r)$ contains $t-1$ summation symbols for each $t > 2$.

(iii) The sequences $\sum_k c_3(n, k, r)$, $n \geq r + 2$, appear in the database [6] under A105483, A105484, A105485, A105486 and A105487, for the fixed values of r : 1, 2, 3, 4, 5.

3. Extension to m-regular partitions

A partition p of $[n]$ contains m-regular t-successions if a block of p contains the arithmetic sequence $x, x + m, \ldots, x + m(t - 1)$, where x is a positive integer. Let $H_t^m(n, k, r)$ denote the set of m-regular k-partitions of $[n]$ containing exactly r (m-regular) t-successions, and let $|H_t^m(n, k, r)| = h_t^m(n, k, r)$. It follows that $h_t^1(n, k, r) = c_t(n, k, r)$. Henceforth we denote partitions by inserting dashes between the members of adjacent blocks. For example 146-2-35 $\in H_2^2(6, 3, 2)$, and 15-2-36-47 $\in H_2^3(7, 4, 2)$.

Theorem 4. *If m is an integer, $m \geq 2$, then*

$$h_2^m(n, k, r) = h_2^{m-1}(n-1, k-1, r); \tag{6}$$

$$h_2^m(n, k, r) = \binom{n-m}{r} S(n-r-m, k-m). \tag{7}$$

Proof. (6) is a consequence of Algorithm 1 described below. By iterating (6) we obtain $h_2^m(n, k, r) = h_2^1(n-m+1, k-m+1, r) = c_2(n-m+1, k-m+1, r)$; and (7) follows from Theorem 1(i). □

Remark. (i) $\lim_{r \to n-k} h_2^m(n, k, r) = \binom{n-m}{k-m}$, i.e., the enumerator of m-regular k-partitions of $[n]$ in which every non-singleton block consists entirely of 2-successions.

(ii) $\sum_k h_2^m(n, k, r) = S(n-m-1, k-m-1) = |H^m(n, k)|$;

$\sum_k h_2^m(n, k, r) = \binom{n-m}{r} B(n-r-m)$, where $B(n)$ is the nth Bell number defined by $B(n) = \sum_k S(n, k)$.

Algorithm 1: Succession Algorithm

We consider a reduction algorithm from [2] for transforming partitions between $H^{m+1}(n+1, k+1)$ and $H^m(n, k)$ which is actually a bijection. Let q be a partition of $[n]$. The graphical representation $G(q)$ of q consists of a line with n vertices $1, 2, \ldots, n$ numbered from left to right. The elements of each block B are joined by a path such that the arc $(x, y) \in G(q)$ if and only if $x, y \in B$ and y covers x. Then the algorithm asserts that a partition $p \in H^{m+1}(n+1, k+1)$ may be reduced to a partition $q \in H^m(n, k)$ by transforming $G(p)$ as follows:

(i) for each $(x, y) \in G(p)$ replace it by $(x, y-1)$;
(ii) delete the vertex $n + 1$.

Table 1
Correspondence between $H^1(4, 2)$ and $H^2(5, 3)$

$H^1(4, 2)$	$H^2(5, 3)$	$W_1((5), 3) = H^2(5, 3)$
123-4	13-24-5	13-25-4
124-3	13-25-4	15-24-3
134-2	14-2-35	14-2-35
14-23	15-24-3	14-25-3
12-34	135-2-4	135-2-4
13-24	14-25-3	13-24-5
1-234	1-24-35	1-24-35

Clearly the above transformation is reversible and $p \in H^{m+1}(n+1, k+1)$ if and only if $q \in H^m(n, k)$.

Denoting $|H^m(n, k)| = h^m(n, k)$, the algorithm gives immediate justification of the following identity

$$h^{m+1}(n+1, k+1) = h^m(n, k). \tag{8}$$

We observe that the bijection preserves 2-successions since it includes the stipulation that $p \in H^{m+1}(n+1, k+1)$ corresponds to $q \in H^m(n, k)$ if and only if every $(m+1)$-regular 2-succession in p maps to an m-regular 2-succession in q such that smaller members of corresponding 2-successions remain fixed.

The proof of (6) follows from (8) and the last remark.

But note that this algorithm fails to preserve t-successions for $t > 2$. The complete set of correspondences between elements of $H^1(4, 2)$ and $H^2(5, 3)$ is shown in the first and second columns of Table 1.

On the other hand, there is a bijection between $H^1(n, k)$ and $H^2(n+1, k+1)$ which transforms members of $H^1(n, k)$ by annihilating 2- and hence all t-successions. The description is a special case of the following general algorithm. The action of the following "succession-killer" algorithm on a partition p of $[n]$ is to annihilate each t- and hence all longer successions in p, and return a partition of $[n+1]$ in which the length of a contained succession is at most $t-1$.

Theorem 5. *Let $W_j((n), k, r)$ denote the set of elements of $W_j(n, k, r)$ which contain the singleton block $\{n\}$, or such that any x $(\neq n)$ in the block containing n implies that the j elements $x+1, x+2, \ldots, x+j$ lie in one other block. Then*

$$|W_j((n+1), k+1)| = S(n, k), \quad j \geq 1,$$

where $W_j((n), k) = \bigcup_{r \geq 0} W_j((n), k, r)$.

Proof. We describe a bijection between $H^1(n, k)$ and $W_j((n+1), k+1)$. A $q \in W_j((n+1), k+1)$ is obtained from $p \in H^1(n, k)$ by transforming p as follows.

(i) if $p \in \bigcup_{r \geq 0} W_j(n, k, r)$, then insert the block $\{n+1\}$ into p;
(ii) else, form the $(k+1)$th block to contain $n+1$ and other elements obtained from p as follows: from each u-succession $x_1, x_2, \ldots, x_u$ $(u \geq j+1)$ select all x_i such that $i \equiv u+1 \pmod{j+1}$. The reverse map is clear. (See Table 2.) $\square$

Table 2
Images of $H^1(4,2)$ in the algorithm of Theorem 5

$H^1(4,2)$	$W_1((5),3)$	$W_2((5),3)$	$W_3((5),3)$
123-4	13-25-4	15-23-4	123-4-5
124-3	15-24-3	124-3-5	124-3-5
134-2	14-2-35	134-2-5	134-2-5
14-23	14-25-3	14-23-5	14-23-5
12-34	135-2-4	12-34-5	12-34-5
13-24	13-24-5	13-24-5	13-24-5
1-234	1-24-35	1-25-34	1-234-5

Remark. (i) If $j = 1$ in Theorem 5, we obtain the bijection $H^1(n,k) \longleftrightarrow H^2(n+1,k+1)$ since $W_1((n+1),k+1) = H^2(n+1,k+1)$. (See the first and third columns of Table 1.)

(ii) If $j \geq n-k+1$, we have $H^1(n,k) = W_j(n,k) = \bigcup_{r\geq 0} W_j(n,k,r)$, which implies $|W_j((n+1),k+1)| = |W_j(n,k)| = S(n,k)$ (see the fourth column of Table 2).

(iii) The block U containing $n+1$ in $q \in W_j((n+1),k+1)$ encodes the number of $(j+1)$-successions in $p \in H^1(n,k)$ such that each $x \in U\backslash\{n+1\}$ represents $y+1$ $(j+1)$-successions in p provided the elements $x-y, x-y+1, \ldots, x+1, x+2, \ldots, x+j$ lie in one block of q, $0 \leq y \leq j$.

4. Set partitions and integer partitions

In this section we present a succession-based classification of set partitions induced by an integer partition which should be compared with the standard one which employs block sizes (see for example [1, p. 215] and [3, p. 205]).

Two elements $y_1, y_2 \in [n]$, $y_1 < y_2$, will be called separated in a partition p if y_1 and y_2 belong to the same block and $y_2 - y_1 \geq 2$, or y_1 and y_2 belong to different blocks of p. With this convention each partition p of $[n]$ possesses a unique succession structure, or equivalently, corresponds to a unique composition $(b_1, b_2, \ldots, b_v)$ of n such that each part b_j represents a b_j-succession, and the greatest member of b_j and the least member of b_{j+1} are separated, $b_j > 0$, $1 \leq j \leq v-1$. For example the succession structure of 145-26-378 is 121112. The succession structure of a partition of $[n]$ can be expressed in the standard partition form $(b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ such that each b_j represents a b_j-succession and appears r_j times, where $0 < b_1 < b_2 < \cdots < b_x$. The last expression is then the corresponding generating partition of n. Conversely, we can start with a partition of n expressed in the standard form and obtain all partitions of $[n]$ of which it is the succession structure. For example, the partitions of [8] generated by the partition $(1^4 2^2)$ of 8 include 145-26-378, 12457-368 and 13-24-56-78.

Let $\pi = (b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ represent a partition of n into v parts. Since a b_j-succession contains $b_j - 1$ 2-successions, the total number of 2-successions in a partition of $[n]$ generated by π is

$$r_1(b_1 - 1) + \cdots + r_x(b_x - 1) = r_1 b_1 + \cdots + r_x b_x - (r_1 + \cdots + r_x) = n - v.$$

Hence we have proved the following proposition.

Proposition 1. *Every partition of n into v parts corresponds to some partition of $[n]$ containing exactly $n-v$ 2-successions.*

Table 3
Classification of partitions of [4] by successions

r	Partitions of 4	Partitions of [4]	$c_2(4, r)$
0	1^4	1-2-3-4, 1-24-3, 14-2-3, 13-2-4, 13-24	5
1	$1^2 2$	1-2-34, 1-23-4, 12-3-4, 14-23, 134-2, 124-3	6
2	13, 2^2	1-234, 123-4, 12-34	3
3	4	1234	1
		$B(4)$	15

It follows that the classification of the partitions of n by numbers of parts implies the classification of the partitions of $[n]$ by numbers of 2-successions, and vice versa. Table 3 illustrates this for $n = 4$, where $c_2(n, r)$ is the number of partitions of $[n]$ containing r 2-successions. Note that $c_2(n, r)$ can be computed directly from

$$c_2(n, r) = \sum_k c_2(n, k, r) = \binom{n-1}{r} B(n - r - 1).$$

Writing $c_2(n, r)$ as $c(n, n - r)B(n - r - 1)$, and recalling that the number of permutations of the sequence of parts of π is $(r_1 + \cdots + r_x)!/(r_1! \ldots r_x!)$, we obtain a refinement of Proposition 1.

Theorem 6. *Every composition of n into v parts corresponds to some partition of $[n]$ containing exactly $n-v$ 2-successions. A partition $(b_1^{r_1}, b_2^{r_2}, \ldots, b_x^{r_x})$ of n generates, by successions, exactly*

$$\frac{(r_1 + \cdots + r_x)!S(r_1 + \cdots + r_x - 1, k - 1)}{r_1! \ldots r_x!}$$

k-partitions of $[n]$, and a total of

$$\frac{(r_1 + \cdots + r_x)!B(r_1 + \cdots + r_x - 1)}{r_1! \ldots r_x!} \tag{9}$$

partitions of $[n]$ (such that each partition of $[n]$ contains a fixed number of 2-successions).

Remark. Theorem 6 can also be derived via a development that employs the class vectors of partitions of $[n]$ (better known as restricted growth functions on $[n]$) [7]. The class vector of a k-partition of $[n]$ is the n-vector $(e_1, \ldots, e_n)$ in which $e_i \in \{1, \ldots, k\}$ and i belongs to block e_i for each i. In this case the succession structure of a partition p of $[n]$ is obtained from its vector by replacing every sequence of x equal and consecutive components by x. Thus the vector of 145-26-378 is (1,2,3,1,1,2,3,3) which gives the composition 111212. Clearly Theorem 6 follows naturally. However, the two approaches do not always give the same structure for a partition.

The number $c_2(n, k, r)$ has a natural dual in the enumerator of k-partitions of $[n]$ with a specified number of separations. Denote the set of k-partitions of $[n]$ with x separations by $S_x(n, k)$. For instance the following partitions are elements of $S_5(8, 3)$: 145-26-378, 1567-24-38. We deduce from Proposition 1 that every k-partition of $[n]$ with x separations contains exactly $n - x - 1$ 2-successions. Hence

$$|S_x(n, k)| = \binom{n-1}{x} S(x, k - 1).$$

It follows that the summands in the standard identities [5]

$$S(n,k)=\sum_{x\geq 0}\binom{n-1}{x}S(x,k-1), \qquad B(n)=\sum_{x\geq 0}\binom{n-1}{x}B(x), \tag{10}$$

correspond to the classification of partitions of $[n]$ by increasing numbers of separations, or equivalently, by decreasing numbers of 2-successions.

Since the occurrence of x separations implies that of $x+1$ separated strings of consecutive integers, we have the following assertion as a bonus.

The number of k-partitions of $[n]$ containing exactly r separate sequences of consecutive integers is given by

$$\binom{n-1}{r-1}S(r-1,k-1).$$

5. Partitions of general sets

We consider the enumeration of the partitions of an arbitrary subset V of $[n]$, where V may have a certain number of separations. Our main tool is Proposition 1.

Theorem 7. *Let V be an ordered N-set of positive integers having x separations, $N>0$ (i.e. V contains x pairs v_i, v_{i+1} such that $v_{i+1}-v_i\geq 2$). The number $c_0((N,x),k)$ of k-partitions of V into subsets of nonconsecutive integers is given by*

$$c_0((N,x),k)=\sum_{j\geq 0}\binom{x}{j}S(N-j-1,k-1). \tag{11}$$

The number $c_2((N,x),k,r)$ of k-partitions of V containing r 2-successions is given by

$$c_2((N,x),k,r)=\binom{N-x-1}{r}\sum_{j\geq 0}\binom{x}{j}S(N-r-j-1,k-1). \tag{12}$$

Note that $c_0((N,0),k)=c_2(N,k,0)$ and $c_2((N,0),k,r)=c_2(N,k,r)$, as expected.

Proof. First renumber the members of V with elements of $[N]$, in size-preserving order. To prove (11) let $C_0(V,k)$ denote the set of the required partitions. Identify each separated pair $v_i, v_{i+1}\in V$ with a succession $u_i, u_{i+1}\in[N]$, $1\leq i\leq x$. Then the latter must appear in some $p\in C_0([N],k)$. But a fixed p can have only j such successions, $j=0,1,\ldots,x$. By Proposition 1 each choice of the j-successions returns $c_2(N-j,k,0)$ partitions. Thus

$$c_0((N,x),k)=\sum_{j\geq 0}\binom{x}{j}c_2(N-j,k,0),$$

and (11) follows from the formula $c_2(n,k,0)=S(n-1,k-1)$. To prove (12) we identify V with a certain composition $\tau=(b_1,b_2,\ldots,b_{x+1})$ which is the succession structure of a partition of $[N]$, and select r of the $N-x-1$ surviving successions. Each choice of r-successions then entails a reduction of the parts of τ as follows:

$$(b_1,b_2,\ldots,b_{x+1})\xrightarrow{-r}(b_1-f_1,b_2-f_2,\ldots,b_{x+1}-f_{x+1}),$$

where $b_i > f_i$ and $(f_1, f_2, \ldots, f_{x+1})$ is a partition of r into $x+1$ nonnegative parts. This returns a set with $N-r$ coded elements, still having x separations. Thus the result follows by applying (11) to the expression

$$c_2((N,x),k,r) = \binom{N-x-1}{r} c_0((N-r,x),k). \quad \square$$

Remark. Assuming the hypothesis of Theorem 7, we are unable to find a formula for the number $c_t((N,x),k,r)$ of k-partitions of V containing r t-successions when $t \geq 3$.

We conclude this section by highlighting a class of refinements of the fundamental Stirling subset-number identity (see the first identity in (10)). Under the hypothesis of Theorem 7, the number of k-partitions of V containing a 2-succession can also be obtained by summing (12) over r, $r \geq 1$, namely

$$\sum_{r\geq 1}\sum_{j\geq 0}\binom{x}{j}\binom{N-x-1}{r} S(N-r-j-1, k-1).$$

Thus an alternative formula for $c_0((N,x),k)$ is

$$c_0((N,x),k) = S(N,k) - \sum_{r\geq 1}\sum_{j\geq 0}\binom{x}{j}\binom{N-x-1}{r} S(N-r-j-1, k-1). \tag{13}$$

Equating (11) with (13) gives the following identity.

$$S(n,k) = \sum_{r+j<n}\binom{v}{j}\binom{n-v-1}{r} S(n-r-j-1, k-1), \tag{14}$$

for each nonnegative integer $v < n$.

The dual of (14) is given by

$$S(n,k) = \sum_{r+j<n}\binom{v}{j}\binom{n-v-1}{r} S(r+j, k-1).$$

The following extensions of the identities may be proved by mathematical induction.

$$\begin{aligned} S(n,k) \;=\; & \sum_{r_1+r_2+\cdots+r_m<n}\binom{v_1}{r_1}\binom{v_2}{r_2}\cdots\binom{v_{m-1}}{r_{m-1}}\binom{n-v_1-v_2-\cdots-v_{m-1}-1}{r_m} \\ & \times S(n-r_1-r_2-\cdots-r_m-1, k-1); \\ S(n,k) \;=\; & \sum_{r_1+r_2+\cdots+r_m<n}\binom{v_1}{r_1}\binom{v_2}{r_2}\cdots\binom{v_{m-1}}{r_{m-1}}\binom{n-v_1-v_2-\cdots-v_{m-1}-1}{r_m} \\ & \times S(r_1+r_2+\cdots+r_m, k-1), \quad r_0 = 0, \end{aligned}$$

for nonnegative integers $v_1, v_2, \ldots, v_{m-1}$, where $v_1 + \cdots + v_{m-1} < n$.

References

[1] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, 1976, Reprinted, Cambridge University Press, Cambridge, 1984, 1998.

[2] W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, Reduction of m-regular noncrossing partitions, European J. Combin. 26 (2) (2005) 237–243.
[3] L. Comtet, Advanced Combinatorics. The Art of Finite and Infinite Expansions, Revised and enlarged edition, D. Reidel Publishing Co., The Netherlands, 1974.
[4] A.O. Munagi, Set partitions with successions and separations, Int. J. Math. Math. Sci. 2005 (2) (2005) 451–463.
[5] J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.
[6] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/njas/sequences/, 2005.
[7] H. Wilf, East Side, West Side . . . an introduction to combinatorial families with Maple programming, December 19, 2000.